Gravitation: **Curvature**

An Introduction to General Relativity

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Notes based on textbook: *Spacetime and Geometry* by S.M. Carroll
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Covariant Derivatives
Parallel Transport and Geodesics
The Riemann Curvature Tensor
Symmetries and Killing Vectors
Maximally Symmetric Spacetimes
Geodesic Deviation
The **metric** defines the **geometry** in a manifold.

Curvature in a manifold depends on the metric, but how.

The form of the metric is strongly dependent on the coordinate system used.

We need a formal definition of curvature.

Curvature plays a central role in **general relativity**: 

\[
\left\{ \text{a measure of local spacetime curvature} \right\} = \left\{ \text{a measure of local matter energy density} \right\}
\]
In $\mathbb{R}^3$, the Gaussian curvature is defined as the product of the principal curvatures; that is, $K = \kappa_1 \kappa_2$. Its value is intrinsic to the surface and does not depend on the embedding.
Recall: The partial derivative of a tensor is not in general a tensor. We need to have a generalization of equations such that $\partial_\nu T^{\mu\nu} = 0$ to be tensorial, so they are invariant under coordinate transformations.

In flat space in inertial coordinates, $\partial_\mu$ is a map from $(k, l)$ tensors to $(k, l + 1)$ that is linear and obey the Leibniz rule.

**Covariant Derivative $\nabla$:**

- $\nabla$ is a map from $(k, l)$ tensor fields to $(k, l + 1)$ tensor fields.
- $\nabla(T + S) = \nabla T + \nabla S$
- Leibniz (product) rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$. 
Covariant Derivative of Vectors

- Consider $v(x^\alpha)$ and $v(x^\alpha + dx^\alpha)$ such that $dx^\alpha = t^\alpha \epsilon$ with $t^\alpha$ defining the direction of the covariant derivative.
- Parallel transport the vector $v(x^\alpha + t^\alpha \epsilon)$ back to the point $x^\alpha$ and call it $v_{\parallel}(x^\alpha)$.
- **Covariant Derivative:**

$$\nabla_t v(x^\alpha) = \lim_{\epsilon \to 0} \frac{v_{\parallel}(x^\alpha) - v(x^\alpha)}{\epsilon}$$

- In a local inertial frame:

$$(\nabla_t v)^\alpha = t^\beta \partial_\beta v^\alpha$$

- Thus,

$$\nabla_\beta v^\alpha = \partial_\beta v^\alpha$$

- Notice: The above expression is not valid in curvilinear coordinates. In general,

$$v_{\parallel}^\alpha(x^\delta) = v^\alpha(x^\delta + \epsilon t^\delta) + \Gamma^\alpha_{\beta \gamma} v^\gamma(x^\delta)(\epsilon t^\beta)$$

<table>
<thead>
<tr>
<th>component changes</th>
<th>basis vector changes</th>
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- Therefore,

$$\nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\beta \gamma} v^\gamma$$
Since the covariant derivative yields tensors,

\[ \nabla_{\mu'} V^{\nu'} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \nabla_\mu V^\nu. \]

thus

\[ \nabla_{\mu'} V^{\nu'} = \partial_{\mu'} V^{\nu'} + \Gamma_{\mu' \lambda'}^{\nu'} \nabla_\lambda V^\lambda \]

and

\[ \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \nabla_\mu V^\nu = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \partial_\mu V^\nu + \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \Gamma_{\mu \lambda}^{\nu \lambda} V^\lambda \]

therefore

\[ \Gamma_{\mu' \lambda'}^{\nu'} \frac{\partial x^\lambda}{\partial x'^\lambda} V^\lambda + \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \frac{\partial}{\partial x^\lambda} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x'^\nu}{\partial x^\nu} \Gamma_{\mu \lambda}^{\nu \lambda} V^\lambda \]

and finally

\[ \Gamma_{\mu' \lambda'}^{\nu'} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial x'^\nu}{\partial x^\nu} \Gamma_{\mu \lambda}^{\nu \lambda} - \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial^2 x'^\nu}{\partial x^\mu \partial x^\lambda}. \]

Notice: the connection coefficients are not the components of a tensor.
Covariant differentiation of 1-forms

A possibility is:

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^\lambda_{\mu \nu} \omega_\lambda$$

To find $\tilde{\Gamma}^\lambda_{\mu \nu}$ we need that $\nabla$

- commutes with contractions: $\nabla_\mu (T^\lambda_{\lambda \rho}) = (\nabla T)^\lambda_{\mu \lambda \rho}$
- reduces to the partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$

Then

$$\nabla_\mu (\omega_\lambda V^\lambda) = (\nabla_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\nabla_\mu V^\lambda)$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \tilde{\Gamma}^\sigma_{\mu \lambda} \omega_\sigma V^\lambda + \omega_\lambda (\partial_\mu V^\lambda) + \omega_\lambda \Gamma^\lambda_{\mu \rho} V^\rho$$

But

$$\nabla_\mu (\omega_\lambda V^\lambda) = \partial_\mu (\omega_\lambda V^\lambda)$$

$$= (\partial_\mu \omega_\lambda) V^\lambda + \omega_\lambda (\partial_\mu V^\lambda)$$

Therefore

$$0 = \tilde{\Gamma}^\sigma_{\mu \lambda} \omega_\sigma V^\lambda + \Gamma^\sigma_{\mu \lambda} \omega_\sigma V^\lambda.$$

Since $\omega_\sigma$ and $V^\lambda$ are completely arbitrary,

$$\tilde{\Gamma}^\sigma_{\mu \lambda} = -\Gamma^\sigma_{\mu \lambda}.$$
Consequently:

Covariant differentiation of Vectors
\[ \nabla_\beta v^\alpha = \partial_\beta v^\alpha + \Gamma^\alpha_{\beta\gamma} v^\gamma \]

Covariant differentiation of 1-forms
\[ \nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \]

Covariant differentiation of general Tensors
\[
\nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} = \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} + \Gamma^{\mu_1}_{\sigma \lambda} T^{\lambda \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} + \Gamma^{\mu_2}_{\sigma \lambda} T^{\mu_1 \lambda \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} + \cdots
\]
Properties of $\Gamma^\alpha_{\mu\nu}$:

- $\Gamma^\alpha_{\mu\nu}$ has $n^3$ components; that is, 64 in 4-dimensions.
- $\Gamma^\alpha_{\mu\nu}$ is called connection because it helps to transport tensors.
- $S^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\mu\nu}$ is a tensor. Recall

$$\Gamma^\nu_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial x'^\nu}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial^2 x'^\nu}{\partial x \partial x^\lambda}$$
$$\tilde{\Gamma}^\nu_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial x'^\nu}{\partial x^\nu} \tilde{\Gamma}^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial^2 x'^\nu}{\partial x \partial x^\lambda}$$

then

$$S^\nu_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\lambda} \frac{\partial x'^\nu}{\partial x^\nu} S^\mu_{\lambda\nu}$$

- If $\Gamma^\nu_{\mu\lambda}$ is a connection, $\Gamma^\nu_{\nu\mu}$ is also a connection. Thus, the torsion tensor is defined by

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = 2\Gamma^\lambda_{[\mu\nu]}.$$  

- A spacetime metric $g_{\mu\nu}$ induces a unique connection if $\Gamma^\alpha_{\mu\nu}$ is (1) torsion-free: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)}$ and (2) metric compatible: $\nabla_\rho g_{\mu\nu} = 0$.

- Metric compatibility implies that

$$\nabla_\rho g^{\mu\nu} = 0$$
$$g_{\mu\lambda} \nabla_\rho V^\lambda = \nabla_\rho (g_{\mu\lambda} V^\lambda) = \nabla_\rho V_{\mu}$$
Christoffel symbols

From metric compatibility:

\[ \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \]
\[ \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \]
\[ \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \]

Subtract the second and third from the first,

\[ \partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2 \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} = 0. \]

Multiply by \( g^{\sigma\rho} \) to get

\[ \Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \]
The Christoffel symbols vanish in flat space in Cartesian coordinates.

The Christoffel symbols do not vanish in flat space in curvilinear coordinates.

For example, if \( ds^2 = dr^2 + r^2 d\theta^2 \), it is not difficult to show that \( \Gamma^r_{\theta\theta} = -r \) and \( \Gamma^\theta_{\theta r} = 1/r \).

At any one point \( p \) in a spacetime \( (M, g_{\mu\nu}) \), it is possible to find a coordinate system for which \( \Gamma^\sigma_{\mu\nu} = 0 \) (recall local flatness).

Very useful property:

\[
\nabla_\mu V^\mu = \partial_\mu V^\mu + \Gamma^\mu_{\mu\lambda} V^\lambda
\]

but

\[
\Gamma^\mu_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} (\partial_\mu g_{\lambda\rho} + \partial_\lambda g_{\rho\mu} - \partial_\rho g_{\mu\lambda})
\]

\[
= \frac{1}{2} g^{\mu\rho} \partial_\lambda g_{\rho\mu} = \frac{1}{2} \partial_\lambda \ln |g| = \frac{1}{\sqrt{|g|}} \partial_\lambda \sqrt{|g|}
\]

then

\[
\nabla_\mu V^\mu = \partial_\mu V^\mu + V^\mu \frac{1}{\sqrt{|g|}} \partial_\mu \sqrt{|g|}
\]

thus

\[
\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu)
\]
As mentioned before, covariant differentiation involves computing how tensor change. However, tensors are maps from vectors and 1-forms to real numbers at a given point. Question: What is behind the changes computed by $\nabla$ acting on tensors? Answer: $\nabla$ gives the instantaneous rate of change of a tensor field in comparison to what the tensor would be if it were parallel transported. The results of parallel transporting a tensor are path dependent.
In flat spacetime, the constancy of a tensor along a curve $x^\mu(\lambda)$ can be stated as:

$$\left(\frac{d}{d\lambda} T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l}\right) = \frac{dx^\sigma}{d\lambda} \partial_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} = 0$$

Define the directional derivative as:

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu$$

Notice that this derivative is a map from $(k, l)$ tensors to $(k, l)$. Thus, the parallel transport condition or equation of parallel transport is defined as:

$$\left(\frac{D}{d\lambda} T\right)^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} \equiv \frac{dx^\sigma}{d\lambda} \nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}_{\nu_1 \nu_2 \cdots \nu_l} = 0$$

For a vector, this equation takes the form

$$\frac{d}{d\lambda} V^\mu + \Gamma^\mu_{\sigma \rho} \frac{dx^\sigma}{d\lambda} V^\rho = 0$$

If the connection $\Gamma^\mu_{\sigma \rho}$ is metric compatible, then

$$\frac{D}{d\lambda} g_{\mu \nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu \nu} = 0$$
Theorem: Given to vectors $V^\mu$ and $W^\mu$ that are parallel-transported along a curve $x^\alpha(\lambda)$, the inner product $g_{\mu\nu} V^\mu W^\nu$ is preserved along this curve.

Proof:

\[
\frac{D}{d\lambda} (g_{\mu\nu} V^\mu W^\nu) = \left( \frac{D}{d\lambda} g_{\mu\nu} \right) V^\mu W^\nu + g_{\mu\nu} \left( \frac{D}{d\lambda} V^\mu \right) W^\nu + g_{\mu\nu} V^\mu \left( \frac{D}{d\lambda} W^\nu \right)
\]

Lemma: The norm of vectors and orthogonality are preserved under parallel transport.
A geodesic generalizes the notion of a straight line in Euclidean space to curved space.

**Definition 1:** A geodesic is the path of shortest distance.

**Definition 2:** A geodesic is the path that parallel transports its own tangent vector.

Definitions 1 and 2 are equivalent if the connection is the Christoffel connection.

From Definition 2, let $dx^\mu / d\lambda$ be the tangent vector to a path $x^\mu(\lambda)$. The condition that $dx^\mu / d\lambda$ be parallel transported is

$$\frac{D}{d\lambda} \frac{dx^\mu}{d\lambda} = 0,$$

or alternatively

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0.$$

This is called the geodesic equation.

Since in Euclidean space in Cartesian coordinates $\Gamma^\mu_{\rho\sigma} = 0$, the geodesic equation becomes $d^2 x^\mu / d\lambda^2 = 0$, which is the equation for a straight line.
From Definition 1, consider a time-like curve \( x^\alpha(\lambda) \) and its corresponding proper time functional

\[
\tau = \int d\tau = \int \sqrt{-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} = \int \left( -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda = \int \sqrt{-f} d\lambda
\]

where we have introduced the following definition: \( f \equiv g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \). The extrema of this functional will give us the shortest-distance path. That is,

\[
\delta \tau = \int \delta \sqrt{-f} d\lambda = -\int \frac{1}{2} (-f)^{-1/2} \delta f d\lambda = 0
\]

Without loss of generality, we can select \( dx^\alpha / d\lambda \) to be the 4-velocity vector \( V^\alpha \); that is, \( \lambda = \tau \) and \( f = -1 \). Therefore the stationary points of \( \tau = \int d\tau \) are equivalent to the stationary points of

\[
l = \frac{1}{2} \int f d\tau = \frac{1}{2} \int g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau
\]

Consider now changes in the proper time under infinitesimal variations of the path,

\[
\begin{align*}
x^\mu &\rightarrow x^\mu + \delta x^\mu \\
g_{\mu\nu} &\rightarrow g_{\mu\nu} + \delta x^\sigma \partial_\sigma g_{\mu\nu}.
\end{align*}
\]

then

\[
\delta l = \frac{1}{2} \int \left( \partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \delta x^\sigma + 2 g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} \right) d\tau
\]
Consider the last term

\[
\int \left( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d(\delta x^\nu)}{d\tau} \right) d\tau = -\int \left( g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \frac{dg_{\mu\nu}}{d\tau} \frac{dx^\nu}{d\tau} \right) \delta x^\nu d\tau
\]

\[
= -\int \left( g_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} + \partial_\sigma g_{\mu\nu} \frac{dx^\sigma}{d\tau} \frac{dx^\nu}{d\tau} \right) \delta x^\nu d\tau
\]

The \( \delta I \) becomes:

\[
\delta I = -\int \left[ g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} \left( -\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] \delta x^\sigma d\tau = 0
\]

which yields

\[
g_{\mu\sigma} \frac{d^2 x^\mu}{d\tau^2} + \frac{1}{2} \left( -\partial_\sigma g_{\mu\nu} + \partial_\nu g_{\mu\sigma} + \partial_\mu g_{\nu\sigma} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0
\]

or

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\mu\nu}^{\sigma} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0
\]
Properties of Geodesics

- The geodesic equation is a generalization of Newton’s law \( f = ma \) for the case \( f = 0 \).
- For the Lorentz force case, in general relativity

\[
\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau} = \frac{q}{m} F^{\mu\nu} \frac{dx^\nu}{d\tau}
\]

- The transformation \( \tau \rightarrow \lambda = a\tau + b \), for some constants \( a \) and \( b \), leaves the geodesic equation invariant. \( \lambda \) is called an affine parameter.
- Notice: The demand that the tangent vector be parallel-transported constrains the parametrization of the curve.
- For a general parametrization,

\[
\frac{d^2 x^\mu}{d\alpha^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\alpha} \frac{dx^\sigma}{d\alpha} = f(\alpha) \frac{dx^\mu}{d\alpha},
\]

where

\[
f(\alpha) = - \left( \frac{d^2 \alpha}{d\lambda^2} \right) \left( \frac{d\alpha}{d\lambda} \right)^{-2}
\]
In a spacetime with Lorentzian metric, the character (timelike/null/spacelike) of the geodesic never changes.

For time-like curves with \( U^\alpha = dx^\alpha / d\tau \) the 4-velocity, the geodesic equation reads \( U^\lambda \nabla_{\lambda} U^\mu = 0 \).

In terms of the 4-momentum \( p^\mu = m U^\mu \) the geodesic equations reads \( p^\lambda \nabla_{\lambda} p^\mu = 0 \).

For a null geodesic the proper time parameter \( \tau \) vanishes. There is no preferred choice of affine parameter in that case. One can for instance pick the affine parameter \( \lambda \) such that \( p^\alpha = dx^\alpha / d\lambda \).

The energy of a particle (time-like or null) is the given by \( E = -p_\mu U^\mu \).
The Riemann Curvature Tensor

Recall that in flat space:

- Parallel-transport around a closed loop leaves a vector unchanged.
- Covariant derivatives of tensors commute.
- Initially parallel geodesics remain parallel.

How do these properties get modified by the presence of curvature and how can we quantify those changes?

Recall also that:

- Parallel-transport of a vector around a closed loop in a curved space will lead to a transformation of the vector.
- The resulting transformation depends on the total curvature enclosed by the loop.

Goal: to have a local description of the curvature at each point. Such description is provided by the Riemann curvature tensor.
Consider the following situation: The parallel-transport of a vector $V^\mu$ around the loop in the figure below.

The change $\delta V^\mu$ experience by $V^\mu$ as it is parallel-transported and returned to the starting point must be

- proportional to $V^\mu$
- depend on $A^\mu$ and $B^\mu$
- anti-symmetric in $A^\mu$ and $B^\mu$ to indicate the direction followed in the loop

thus

$$\delta V^\rho = (\delta a)(\delta b) A^\nu B^\mu R^\rho_{\sigma \mu \nu} V^\sigma$$

where $R^\rho_{\sigma \mu \nu}$ is a $(1, 3)$ tensor known as the Riemann or curvature tensor. Notice:

$$R^\rho_{\sigma \mu \nu} = -R^\rho_{\sigma \nu \mu}$$
Commutator of two covariant derivatives: it measures the difference between parallel transporting the tensor first one way and then the other, versus the opposite ordering.

That is:

\[
[\nabla_\mu, \nabla_\nu] V^\rho = \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho \\
= \partial_\mu (\nabla_\nu V^\rho) - \Gamma^\lambda_{\mu \nu} \nabla_\lambda V^\rho + \Gamma^\rho_{\mu \sigma} \nabla_\nu V^\sigma - (\mu \leftrightarrow \nu) \\
= \partial_\mu \partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu \sigma}) V^\sigma + \Gamma^\rho_{\nu \sigma} \partial_\mu V^\sigma - \Gamma^\lambda_{\mu \nu} \partial_\lambda V^\rho - \Gamma^\lambda_{\mu \nu} \Gamma^\rho_{\lambda \sigma} V^\sigma \\
+ \Gamma^\rho_{\mu \sigma} \partial_\nu V^\sigma + \Gamma^\rho_{\mu \sigma} \Gamma^\sigma_{\nu \lambda} V^\lambda - (\mu \leftrightarrow \nu) \\
= (\partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\lambda_{\nu \lambda} \Gamma^\rho_{\mu \sigma}) V^\sigma - 2\Gamma^\lambda_{[\mu \nu]} \nabla_\lambda V^\rho \\
= \mathbf{R}^\rho_{\sigma \mu \nu} V^\sigma - T^\lambda_{\mu \nu} \nabla_\lambda V^\rho 
\]

where

**Riemann Tensor**

\[
\mathbf{R}^\rho_{\sigma \mu \nu} = \partial_\mu \Gamma^\rho_{\nu \sigma} - \partial_\nu \Gamma^\rho_{\mu \sigma} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\mu \sigma} 
\]
Important to notice:

- The antisymmetry of $R^\rho_{\sigma\mu\nu}$ in $\mu\nu$ is obvious.
- The derivation depends only on the connection (no mention of the metric was made).
- Thus, the definition is true for any connection, whether or not it is metric compatible or torsion free.
- The action of $[\nabla_\rho, \nabla_\sigma]$ can be generalized to a tensor of arbitrary rank:

$$[\nabla_\rho, \nabla_\sigma]X^{\mu_1\cdots\mu_k\nu_1\cdots\nu_l} = - T^\lambda_\rho_\sigma \nabla_\lambda X^{\mu_1\cdots\mu_k\nu_1\cdots\nu_l} + R^{\mu_1}_\rho_\sigma X^{\lambda\mu_2\cdots\mu_k\nu_1\cdots\nu_l} + R^{\mu_2}_\rho_\sigma X^{\mu_1\lambda\cdots\mu_k\nu_1\cdots\nu_l} + \cdots - R^\lambda_{\nu_1\rho\sigma} X^{\mu_1\cdots\mu_k\lambda\nu_2\cdots\nu_l} - R^\lambda_{\nu_2\rho\sigma} X^{\mu_1\cdots\mu_k\nu_1\lambda\cdots\nu_l} - \cdots$$

One can view the torsion tensor and the curvature tensors as

- $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$  
- $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$

where $\nabla_X = X^\mu \nabla_\mu$. That is

$$R^\rho_{\sigma\mu\nu} X^\mu Y^\nu Z^\sigma = X^\lambda \nabla_\lambda (Y^\eta \nabla_\eta Z^\rho) - Y^\lambda \nabla_\lambda (X^\eta \nabla_\eta Z^\rho) - (X^\lambda \partial_\lambda Y^\eta - Y^\lambda \partial_\lambda X^\eta) \nabla_\eta Z^\rho$$
Theorem:
- If the components of the metric are constant in some coordinate system, the Riemann tensor will vanish.
- If the Riemann tensor vanishes we can always construct a coordinate system in which the metric components are constant.

Proof:
- Part 1: If in some coordinate system \( \partial_\sigma g_{\mu\nu} = 0 \), then \( \Gamma^\rho_{\mu\nu} = 0 \) and \( \partial_\sigma \Gamma^\rho_{\mu\nu} = 0 \); thus \( R^\rho_{\sigma\mu\nu} = 0 \).
- Part 2: see textbook
How many independent components does the Riemann tensor have?

- In principle, it has $n^4$ independent components in $n$-dimensions.
- The anti-symmetry in the last two indices implies there are only $n(n-1)/2$ independent values these last two indices can take on, there are $n^3(n-1)/2$ independent components.

Consider $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda_{\;\sigma\mu\nu}$ in Riemann:

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda}(\partial_\mu \Gamma^\lambda_{\nu\sigma} - \partial_\nu \Gamma^\lambda_{\mu\sigma})$$
$$= \frac{1}{2} g_{\rho\lambda} g^{\lambda\tau}(\partial_\mu \partial_\nu g_{\sigma\tau} + \partial_\mu \partial_\tau g_{\sigma\nu} - \partial_\nu \partial_\mu g_{\sigma\tau} - \partial_\nu \partial_\sigma g_{\tau\mu} + \partial_\nu \partial_\tau g_{\mu\sigma})$$
$$= \frac{1}{2}(\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\nu \partial_\rho g_{\mu\sigma})$$

then

$$R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$$
$$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$$
$$R_{\rho[\sigma\mu\nu]} = 0$$

Since these are tensorial equations, they are also true in any coordinate system.

- There are $1/12 n^2(n^2 - 1)$ independent components in the Riemann tensor. In 4-dimensions, there are 20 independent components.
The Bianchi Identity

Consider the covariant derivative of the Riemann tensor, evaluated in Riemann normal coordinates:

\[
\nabla_\lambda R_{\rho\sigma\mu\nu} = \partial_\lambda R_{\rho\sigma\mu\nu} = \frac{1}{2} \partial_\lambda \left( \partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\nu \partial_\rho g_{\mu\sigma} \right).
\]

and consider

\[
\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = \frac{1}{2} \left( \partial_\lambda \partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\lambda \partial_\mu \partial_\rho g_{\nu\sigma} - \partial_\lambda \partial_\nu \partial_\sigma g_{\rho\mu} + \partial_\lambda \partial_\nu \partial_\rho g_{\mu\sigma} \\

+ \partial_\rho \partial_\mu \partial_\lambda g_{\sigma\nu} - \partial_\rho \partial_\mu \partial_\sigma g_{\nu\lambda} - \partial_\rho \partial_\nu \partial_\lambda g_{\sigma\mu} + \partial_\rho \partial_\nu \partial_\sigma g_{\mu\lambda} \\

+ \partial_\sigma \partial_\mu \partial_\rho g_{\lambda\nu} - \partial_\sigma \partial_\mu \partial_\lambda g_{\rho\nu} - \partial_\sigma \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\nu \partial_\lambda g_{\rho\mu} \right) = 0.
\]

Since \( R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} \) then

Bianchi identity

\[
\nabla_\lambda R_{\rho\sigma\mu\nu} = 0.
\]
The Ricci Tensor

\[ R_{\mu \nu} = R^{\lambda}_{\mu \lambda \nu} . \]

Because of \( R_{\rho \sigma \mu \nu} = R_{\mu \nu \rho \sigma} \)

\[ R_{\mu \nu} = R_{\nu \mu} , \]

Also

\[ R = R^{\mu}_{\mu} = g^{\mu \nu} R_{\mu \nu} . \]
The Weyl Tensor

\[ C_{\rho \sigma \mu \nu} = R_{\rho \sigma \mu \nu} - \frac{2}{(n-2)} \left( g_{\rho [\mu} R_{\nu ] \sigma} - g_{\sigma [\mu} R_{\nu ] \rho} \right) + \frac{2}{(n-1)(n-2)} R g_{\rho [\mu} g_{\nu ] \sigma} . \]

Notice:

- The Ricci tensor and the Ricci scalar contain information about “traces” of the Riemann tensor. The Weyl tensor is the Riemann tensor with “all of its contractions removed.”
- All possible contractions of \( C_{\rho \sigma \mu \nu} \) vanish, while it retains the symmetries of the Riemann tensor:

\[
\begin{align*}
C_{\rho \sigma \mu \nu} &= C_{[\rho \sigma] [\mu \nu]} , \\
C_{\rho \sigma \mu \nu} &= C_{\mu \nu \rho \sigma} , \\
C_{\rho [\sigma \mu \nu]} &= 0 .
\end{align*}
\]

- The Weyl tensor is only defined in three or more dimensions, and in three dimensions it vanishes identically.
- For \( n \geq 4 \) it satisfies a version of the Bianchi identity,

\[
\nabla^\rho C_{\rho \sigma \mu \nu} = -2 \frac{(n-3)}{(n-2)} \left( \nabla_{[\mu} R_{\nu ] \sigma} + \frac{1}{2(n-1)} g_{\sigma [\nu} \nabla_{\mu]} R \right) .
\]

- The Weyl tensor is invariant under conformal transformations. That is, \( C_{\rho \sigma \mu \nu} \) for some metric \( g_{\mu \nu} \) and \( \tilde{C}_{\rho \sigma \mu \nu} \) for a metric \( \tilde{g}_{\mu \nu} = \Omega^2(x) g_{\mu \nu} \) are the same. So the Weyl tensor is a.k.a. the conformal tensor.
The Einstein Tensor

Contract twice the Bianchi identity

\[ \nabla_\lambda R_{\rho \sigma \mu \nu} + \nabla_\rho R_{\sigma \lambda \mu \nu} + \nabla_\sigma R_{\lambda \rho \mu \nu} = 0 \]

to get

\[ 0 = g^{\nu \sigma} g^{\mu \lambda} (\nabla_\lambda R_{\rho \sigma \mu \nu} + \nabla_\rho R_{\sigma \lambda \mu \nu} + \nabla_\sigma R_{\lambda \rho \mu \nu}) = \nabla^\mu R_{\rho \mu} - \nabla_\rho R + \nabla^\nu R_{\rho \nu}, \]

or

\[ \nabla^\mu R_{\rho \mu} - \frac{1}{2} \nabla_\rho R = 0 \]

Define

**Einstein Tensor**

\[ G_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu}, \]

to get

**Contracted Bianchi Identity**

\[ \nabla^\mu G_{\mu \nu} = 0 \]
Consider the two-sphere, with metric
\[ ds^2 = a^2(d\theta^2 + \sin^2 \theta \, d\phi^2) , \]
where \( a \) the radius of the sphere. Then coefficients are
\[ \Gamma^\theta_{\phi\phi} = - \sin \theta \cos \theta \]
\[ \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot \theta . \]
The only non-vanishing component of the Reimann tensor is
\[
R^\theta_{\phi\theta\phi} = \partial_\theta \Gamma^\theta_{\phi\phi} - \partial_{\phi} \Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda} \Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda} \Gamma^\lambda_{\theta\phi} = (\sin^2 \theta - \cos^2 \theta) - (0) + (0) - (- \sin \theta \cos \theta)(\cot \theta) = \sin^2 \theta .
\]
Thus
\[
R_{\theta\phi\theta\phi} = g_{\theta\lambda} R^\lambda_{\phi\theta\phi} = g_{\theta\phi} R^\theta_{\phi\theta\phi} = a^2 \sin^2 \theta .
\]
From \( R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\mu\beta\nu} \) one gets
\[
R_{\theta\theta} = g^{\phi\phi} R_{\phi\theta\phi\theta} = 1
\]
\[ R_{\theta\phi} = R_{\phi\theta} = 0 \]
\[ R_{\phi\phi} = g^{\theta\theta} R_{\theta\phi\theta\phi} = \sin^2 \theta . \]
and the Ricci scalar
\[ R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = \frac{2}{a^2} . \]
The Ricci scalar for a two-dimensional manifold completely characterizes the curvature.
For this example it is a constant over the two-sphere.
The manifold is maximally symmetric.
In any number of dimensions, the curvature of a maximally symmetric space satisfies (for some constant $a$)

$$R_{\rho\sigma\mu\nu} = a^{-2}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}),$$
A manifold $M$ possesses a symmetry if the geometry is invariant under a certain transformation that maps $M$ into itself.

Isometry: a symmetry of the metric.

Example: 4-dimensional Minkowski space-time. The metric

$$ds^2 = \eta_{\mu\nu} dx^\mu \ dx^\nu$$

has the symmetries

\begin{align*}
x^\mu & \rightarrow x^\mu + a^\mu & \text{translations} \\
x^\mu & \rightarrow \Lambda^\mu_{\ \nu} x^\nu & \text{Lorentz transformation}
\end{align*}

Notice:

$$\partial_{\sigma*} g_{\mu\nu} = 0 \quad \Rightarrow \quad x^{\sigma*} \rightarrow x^{\sigma*} + a^{\sigma*}$$ is a symmetry
Recall the geodesic equation

\[ 0 = p^\nu \nabla_\nu p_\mu \]

\[ = p^\nu \partial_\nu p_\mu - \Gamma^\delta_{\mu\nu} p^\nu p_\delta = m \frac{dx^\nu}{d\tau} \partial_\nu p_\mu - \Gamma^\delta_{\mu\nu} p^\nu p_\delta \]

\[ = m \frac{dp_\mu}{d\tau} - \frac{1}{2} g^{\delta\lambda} \left( \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) p^\nu p_\delta \]

\[ = m \frac{dp_\mu}{d\tau} - \frac{1}{2} \left( \partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right) p^\nu p^\lambda \]

\[ = m \frac{dp_\mu}{d\tau} - \frac{1}{2} \partial_\mu g_{\nu\lambda} p^\nu p^\lambda \]

therefore

4-Momentum Conservation

\[ \partial_{\sigma^*} g_{\mu\nu} = 0 \quad \text{along a geodesic} \quad \Rightarrow \quad \frac{dp_{\sigma^*}}{d\tau} = 0 \]
Consider the following coordinate transformation on the 4-dimensional Minkowski metric: \( z = \dot{x} \). Then

\[
\text{ds}^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu = -dt^2 + (1 + z^2)dx^2 + dy^2 + x^2 \, d\hat{z}^2 + 2x\dot{z} \, dx \, d\hat{z}
\]

Where did the translational symmetries go? We need a method to define symmetries invariant under coordinate transformations.

- Define \( K = \partial_{\sigma^*} \) such that \( \partial_{\sigma^*} g_{\mu\nu} = 0 \).
- This is equivalent to \( K^\mu = (\partial_{\sigma^*})^\mu = \delta^\mu_{\sigma^*} \).
- If \( K \) is the generator of an isometry, then \( p_{\sigma^*} = K^\nu p_\nu = K_\nu p^\nu \) is invariant along the \( K \)-direction.
- That is

\[
\frac{dp_{\sigma^*}}{d\tau} = 0 \quad \iff \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0
\]

Expanding the last equation

\[
0 = p^\mu \nabla_\mu (K_\nu p^\nu) \\
= p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu \\
= p^\mu p^\nu \nabla_\mu K_\nu = p^\mu p^\nu \nabla_{(\mu} K_{\nu)}
\]

Therefore

\[
\nabla_{(\mu} K_{\nu)} = 0 \quad \iff \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0
\]

**Killing’s Equation**

\[
\nabla_{(\mu} K_{\nu)} = 0
\]
Theorem: If a vector $K$ is a Killing vector, it is always possible to find a coordinate system in which $K = \partial_{\sigma^*}$. If there are more than one Killing vector. In general it is not possible to find coordinates for which all vectors are of the form $K = \partial_{\sigma^*}$.

Notice: For a Killing vector, the Riemann equation

$$\nabla_{[\mu} \nabla_{\nu]} K^\rho = R^\rho{}_{\sigma \mu \nu} K^\sigma$$

takes the form

$$\nabla_{\mu} \nabla_{\nu} K^\rho = R^\rho{}_{\nu \mu \sigma} K^\sigma$$

thus

$$\nabla_{\mu} \nabla_{\nu} K^\mu = R_{\nu \sigma} K^\sigma$$

From the contracted Bianchi identity

$$0 = K^\nu \left( \nabla^\mu R_{\mu \nu} - \frac{1}{2} \nabla_{\nu} R \right)$$

$$= 2 K^\nu \nabla^\mu R_{\mu \nu} - K^\mu \nabla_{\mu} R$$

$$= 2 \nabla^\mu (K^\nu R_{\mu \nu}) - 2 R_{\mu \nu} \nabla^\mu K^\nu - K^\mu \nabla_{\mu} R$$

$$= 2 \nabla^\mu (\nabla_\lambda \nabla_\mu K^\lambda) - 2 R_{\mu \nu} \nabla^{(\mu} K^{\nu)} - K^\mu \nabla_{\mu} R$$

$$= 2 \nabla^\mu \nabla^\lambda \nabla_\mu K_\lambda - 2 R^{\mu \nu} \nabla^{(\mu} K^{\nu)} - K^\mu \nabla_{\mu} R$$

$$= K^\mu \nabla_{\mu} R$$

Pablo Laguna  Gravitation: Curvature
Killing vectors in flat space

Translations:

\[ X^{\mu} = (1, 0, 0) \]
\[ Y^{\mu} = (0, 1, 0) \]
\[ Z^{\mu} = (0, 0, 1) \]

Rotations:

\[ R^{\mu} = (-y, x, 0) \]
\[ S^{\mu} = (z, 1, -x) \]
\[ T^{\mu} = (0, -z, y) \]
The notion of parallel does not extend naturally from flat to curved spaces. Instead, construct a one-parameter family of non-intersecting geodesics, $\gamma_s(t)$; namely, for each $s \in \mathbb{R}$ there is a geodesic $\gamma_s$ with affine parameter $t$.

The collection $\gamma_s(t)$ defines a smooth two-dimensional surface. Chose $s$ and $t$ to be the coordinates in this surface. The entire surface is the set of points $x^\mu(s, t) \in M$.

There are two natural vector fields:

$$T^\mu = \frac{\partial x^\mu}{\partial t}$$

$$S^\mu = \frac{\partial x^\mu}{\partial s}$$

tangent vectors

deviation vectors
Define

\[ V^\mu = (\nabla_T S)^\mu = T^\rho \nabla_\rho S^\mu \] relative velocity of geodesics

\[ a^\mu = (\nabla_T V)^\mu = T^\rho \nabla_\rho V^\mu \] relative acceleration of geodesics

Since \( S \) and \( T \) are basis vectors adapted to a coordinate system, \([S, T] = 0\)

Since we are considering vanishing torsion, \([S, T] = 0\) implies that

\[ S^\rho \nabla_\rho T^\mu - T^\rho \nabla_\rho S^\mu = 0 \]

Thus

\[ a^\mu = T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \]

\[ = T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) \]

\[ = (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu \]

\[ = (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\sigma \nabla_\rho T^\mu + R^\mu_{\nu \rho \sigma} T^\nu) \]

\[ = (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu + R^\mu_{\nu \rho \sigma} T^\nu T^\rho S^\sigma \]

\[ = R^\mu_{\nu \rho \sigma} T^\nu T^\rho S^\sigma . \]

Therefore

Geodesic deviation equation

\[ a^\mu = \frac{D^2}{dt^2} S^\mu = R^\mu_{\nu \rho \sigma} T^\nu T^\rho S^\sigma , \]